ON THE INTEGRATION OF POISSON MANIFOLDS, LIE ALGEBROIDS, AND COISOTROPIC SUBMANIFOLDS

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ABSTRACT. In recent years methods for the integration of Poisson manifolds and of Lie algebroids have been proposed, the latter being usually presented as a generalization of the former. In this note it is shown that the latter method is actually related to (and may be derived from) a particular case of the former if one regards dual of Lie algebroids as special Poisson manifolds. The core of the proof is the fact, discussed in the second part of this note, that coisotropic submanifolds of a (twisted) Poisson manifold are in one-to-one correspondence with possibly singular Lagrangian subgroupoids of source-simply-connected (twisted) symplectic groupoids.

1. INTRODUCTION

The "infinitesimal form" $\mathsf{Lie}(G)$ of a Lie groupoid G (viz., the bundle of vectors tangent to the source-fibers restricted to the manifold of units) is naturally equipped with a Lie algebroid structure (we will recall definitions and basic facts in Sect. 2). On the other hand, there exist Lie algebroids that do not arise this way. The special ones in the image of the Lie-functor are called integrable. When a Lie algebroid is integrable, then there is a unique source-simply-connected (ssc) Lie groupoid, up to isomorphisms [13].

The cotangent bundle of a Poisson manifold may be given the structure of a Lie algebroid with anchor map induced from the Poisson bivector field and the Lie bracket on 1-forms given by Koszul [10]. When this Lie algebroid is integrable, one says that the Poisson manifold is integrable. The ssc Lie groupoid can in this case be endowed with a multiplicative symplectic form and is called a symplectic groupoid.

The basic example is the cotangent bundle T^*G of a Lie group G as a Lie groupoid over the Poisson manifold $(\mathsf{Lie}(G))^*$.

As we will recall in Sect. 3, a method was introduced in [3] to integrate Poisson manifolds to symplectic groupoids by symplectic reduction from an infinite-dimensional manifold (the cotangent bundle of the path space of the Poisson manifold). The symplectic quotient is in general singular, and actually this happens iff the Poisson manifold is not integrable. The authors of [15] and [7] independently observed that the above method allows for a generalization to any Lie algebroid. The main contribution of [7] is then to use this construction to characterize integrable Lie algebroids in terms of an if-and-only-if criterion. See also [8] for a general discussion of obstructions to integrability in the context of Poisson manifolds.

The aim of this note is to show that the generalization of [15, 7] may as well be seen as a particular case of the previous construction in [3]. The main observation is that the dual bundle A^* of a Lie algebroid A is naturally a Poisson manifold. Moreover, if $A = \mathsf{Lie}(G)$, then T^*G can be given a symplectic groupoid structure for the Poisson manifold A^* [6]. As we will recall at the end of 2.4, it follows from classical results— [11], [2]—that the integrability of the Lie algebroid A is equivalent to the integrability of the Poisson manifold A^*. We want to show that also the integration methods of [7, 15] and of [3] are equivalent. On the one hand, [3] may be recovered from [7, 15] as a particular case (by looking at the Lie algebroid of a Poisson manifold). On the other hand, we show in Sect. 4 that [7, 15] may be obtained from [3] by observing that to a Lie algebroid A one may naturally associate a Lagrangian submanifold of the cotangent bundle of the path space of the Poisson manifold A^*. Symplectic reduction then associates to it a Lagrangian submanifold of the symplectic groupoid T^*G of A^* that turns out to be (isomorphic to) the Lie groupoid G of A.

The above construction turns out to be a special case of a more general correspondence between coisotropic submanifolds of a Poisson manifold and Lagrangian Lie subgroupoids of its symplectic groupoid. One direction of this correspondence (from Lagrangian subgroupoids to coisotropic submanifolds) follows from results in [19]. The other direction (the "integration") is proved in Sect. 5.

Twisted Poisson manifolds [16] have recently become popular. The results of Sect. 5 extend to the twisted case, as we discuss in Sect. 6.

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2. Preliminaries

In this Section we review some basic notions and fix notations.

2.1. **Lie algebroids.** A Lie algebroid $(A, B, \rho, [\ ,\ ])$ is a vector bundle A over a manifold B together with a bundle map $\rho \colon A \to TB$ (the anchor) and a Lie bracket $[\ ,\ ]$ on the real vector space $\Gamma(A)$ of sections of A satisfying the following compatibility condition:
(2.1)
$$[X, fY] = f[X, Y] + \mathrm{L}_{\rho_* X} f\, Y, \qquad X, Y \in \Gamma(A),\ f \in C^\infty(B),$$

where $\rho_* \colon \Gamma(A) \to \mathfrak{X}(B)$ is the induced map of sections and L denotes the Lie derivative. It follows that ρ_* is a morphism of Lie algebras. We recall some examples of Lie algebroids:

(1) Any vector bundle with trivial anchor and Lie bracket.
(2) A Lie algebra regarded as a vector bundle over a point.
(3) The tangent bundle TB of a manifold B with the usual Lie bracket of vector fields and ρ the identity map.
(4) An involutive subbundle A of the tangent bundle TB with the usual Lie bracket and ρ the inclusion map.

A Lie algebroid structure on A allows one to define a differential δ on the complex $\Gamma(\Lambda^\bullet A^*)$, where A^* denotes the dual bundle, by the rules
$$\delta f := \rho^* \mathrm{d} f, \quad f \in C^\infty(B) = \Gamma(\Lambda^0 A^*)$$
and
$$\langle \delta\alpha, X \wedge Y \rangle := - \langle \alpha, [X, Y] \rangle + \langle \delta\langle \alpha, X \rangle, Y \rangle - \langle \delta\langle \alpha, Y \rangle, X \rangle,$$
$$X, Y \in \Gamma(A),\ \alpha \in \Gamma(A^*),$$

where $\rho^* \colon \Omega^1(B) \to \Gamma(A^*)$ is the transpose of ρ_* and $\langle\ ,\ \rangle$ is the canonical pairing of sections of A^* and A.

If we have a bundle map ϕ between vector bundles $A \to B$ and $\widetilde{A} \to \widetilde{B}$ with base map φ, we may define the pullback $\phi^* \colon \Gamma(\Lambda^\bullet \widetilde{A}^*) \to \Gamma(\Lambda^\bullet A^*)$ as the algebra homomorphism which is φ^* in degree zero and the induced map[1] of sections $\phi^* \colon \Gamma(\widetilde{A}^*) \to \Gamma(A^*)$ in degree one. If A

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[1] As usual, this is defined by setting $\phi^*(\sigma)(b) = \phi_b^* \sigma(\varphi(b))$, $\sigma \in \Gamma(\widetilde{A}^*)$, $b \in B$, where ϕ_b^* is the transpose of the linear map $\phi_b \colon A_b \to \widetilde{A}_{\varphi(b)}$.

and $\widetilde{A}$ are Lie algebroids, a bundle map ϕ is said to be a **morphism** if ϕ^* is a chain map w.r.t. the corresponding differentials.

If we choose local coordinates $\{b^i\}_{i=1,\dots,\dim B}$ on a trivializing chart U and pick a basis $\{e^\mu\}_{\mu=1,\dots,\operatorname{rank} A}$ on the fiber (which we also regard as a basis of constant sections of $A_{|_U}$), we may introduce the anchor functions $\rho^{\mu i}$ and the structure functions $f_\sigma^{\mu\nu}$ by the equations

$$\rho_*(e^\mu)(b) = \rho^{\mu i}(b)\,\frac{\partial}{\partial b^i}, \qquad [\,e^\mu,\,e^\nu\,](b) = f_\sigma^{\mu\nu}(b)\,e^\sigma,$$

where summation over repeated indices is understood. The compatibility condition (2.1) corresponds locally to PDEs to be satisfied by the anchor and structure functions.

2.2. Lie groupoids.

A **groupoid** is a small category where all morphisms are invertible. Explicitly, we have a set G of morphisms and a set B of objects together with structure maps satisfying certain axioms. First we have the surjective **source** and **target** maps $s, t\colon G \to B$ and the **identity** bisection $\epsilon\colon B \hookrightarrow G$. Then we have the multiplication $m\colon G^{(2)} \to G$, with $G^{(2)} := \{(u,v) \in G \times G : s(u) = t(v)\}$; as a shorthand notation we will also write uv instead of $m(u,v)$. Finally, we have an **inverse** map $G \to G$, $u \mapsto u^{-1}$. The axioms to be satisfied are

$$s(uv) = s(v), \quad t(uv) = t(u), \quad \epsilon(b)v = v, \quad u\epsilon(b) = u, \quad (uv)w = u(vw),$$
$$s(u^{-1}) = t(u), \quad t(u^{-1}) = s(u), \quad uu^{-1} = \epsilon(t(u)), \quad u^{-1}u = \epsilon(s(u)),$$

for all $u, v, w \in G$ and $b \in B$ for which the above expressions are meaningful. The set G is usually referred to as the groupoid, while the set B is the base. To denote a groupoid G with base B, we will often use the notation $G \rightrightarrows B$. A **morphism** between groupoids $G \rightrightarrows B$ and $\widetilde{G} \rightrightarrows \widetilde{B}$ is just a functor, i.e., a pair of maps $\Phi\colon G \to \widetilde{G}$ and $\varphi\colon B \to \widetilde{B}$ compatible with the structure maps s, t, ϵ, m.

For a groupoid to be a **Lie groupoid** [14], one first requires that G should be a (possibly non Hausdorff) manifold, that B should be a Hausdorff manifold and that the source, target, identity and inverse maps should be smooth. One further requires that the source (or equivalently the target) map should be a submersion. This makes $G^{(2)}$ into a manifold, too, and one eventually requires that the multiplication map should also be smooth. One says that a Lie groupoid is **source-simply-connected (ssc)** if the s-fibers are connected and simply connected. A **morphism** of Lie groupoids is a smooth morphism of the underlying groupoids.

Here are some examples of Lie groupoids:

   (1) G a vector bundle over B with $s = t = $ the projection, ϵ the
       zero section, multiplication $(b, a)(b, a') = (b, a + a')$ and inverse
       $(b, a)^{-1} = (b, -a)$.
   (2) G a Lie group and B a point.
   (3) $G = B \times B$ with s and t the two projections, ϵ the diagonal
       map, multiplication $(b, b')(b', b") = (b, b")$ and inverse $(b, b')^{-1} = (b', b)$.

The vector bundle $\ker(\mathrm{d}s)_{|_{\epsilon(B)}}$ has a natural structure of a Lie algebroid over B with anchor $\mathrm{d}t$ and Lie bracket induced by the multiplication. A morphism of Lie groupoids induces a morphism of the corresponding Lie algebroids by taking its differential at the identity sections.

We will denote by $\mathsf{Lie}(G)$ the Lie algebroid of the Lie groupoid G. The Lie algebroids of the examples 1), 2) and 3) above are the ones described in the previous subsection with the same numbers. It is a fundamental fact that not all Lie algebroids arise in this way. Those which do are usually called integrable. Other fundamental facts of the theory of Lie groups generalize to Lie groupoids:

> **Lie I:** Let $A = \mathsf{Lie}(G)$ and let $\widetilde{A}$ be a Lie subalgebroid of A. Then there is a Lie subgroupoid of G with $\widetilde{A}$ as its Lie algebroid.
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> **Lie II:** Let G and $\widetilde{G}$ be Lie groupoids, and let A and $\widetilde{A}$ be the corresponding Lie algebroids. If G is ssc, for every morphism $\phi \colon A \to \widetilde{A}$ there is a unique morphism $\Phi \colon G \to \widetilde{G}$ that induces ϕ.

For more information on Lie groupoids and Lie algebroids, see [11], [2] and [13].

2.3. **Poisson manifolds.** A Poisson manifold (M, π) is a smooth manifold M together with a bivector field (i.e., a section of $\Lambda^2 TM$) π such that the bracket of functions $\{ f , g \} := \pi(\mathrm{d}f, \mathrm{d}g)$ satisfies the Jacobi identity. Examples of Poisson manifolds are strong[2] symplectic manifolds (with the usual Poisson bracket of functions) and dual spaces of Lie algebras (with linear Poisson bracket induced by the Lie bracket). In this paper we will only consider finite-dimensional Poisson manifolds.

Lie algebroids and Poisson manifolds are strict relatives as we describe in the following.

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[2]A symplectic form on M is a closed, nondegenerate 2-form. If it induces an isomorphism $TM \to T^*M$, as always happens in finite dimensions, and not just a monomorphism, it is called strong; otherwise it is called weak.

2.3.1. *The Lie algebroid of a Poisson manifold.* To each Poisson manifold M one may associate a Lie algebroid structure on T^*M with anchor the bundle map $\pi^\#\colon T^*M \to TM$ defined by $\pi^\#(x)(\sigma) = \pi(x)(\sigma, \bullet)$, with $x \in M$, $\sigma \in T^*_x M$, and the Koszul bracket which on exact forms is defined by $[\,\mathrm{d}f\,,\,\mathrm{d}g\,] := \mathrm{d}\{\,f\,,\,g\,\}$ and is extended to arbitrary 1-forms using the rule (2.1).[3] We will denote the Lie algebroid of the Poisson manifold (M, π) by $T^*_\pi M$. One says that a Poisson manifold is integrable if its Lie algebroid is.

In local coordinates $\{x^i\}_{i=1,\dots,\dim M}$, we have the local basis $\mathrm{d}x^i$ of constant 1-forms w.r.t. which the anchor functions are the components π^{ij} and the structure functions are the partial derivatives $\partial_i \pi^{rs}$.

2.3.2. *Lie algebroids as Poisson manifolds.* If $(A, B, \rho, [\ ,\ ])$ is a Lie algebroid, the dual bundle A^* has a natural Poisson structure. This is defined first on functions that are constant on the fibers (i.e., functions on B) or linear on the fibers (i.e., sections of A):

$$\{\,F\,,\,G\,\} = \begin{cases} 0 & \text{if } F, G \in C^\infty(B), \\ \mathrm{L}_{\rho_* F} G & \text{if } F \in \Gamma(A) \text{ and } G \in C^\infty(B), \\ [\,F\,,\,G\,] & \text{if } F, G \in \Gamma(A). \end{cases}$$

The bracket is then extended to functions that are polynomial on the fibers (i.e., sections of the symmetric powers of A) as a skew-symmetric biderivation and finally to all smooth functions by completion.

If we choose local coordinates $\{b^i\}_{i=1,\dots,\dim B}$ and $\{\alpha^\mu\}_{\mu=1,\dots,\mathrm{rank}\,A}$ on A^*, then the local coordinate expression of the bivector field corresponding to the above Poisson structure is

$$(2.2) \qquad \pi(b, \alpha) = \alpha^\sigma f^{\mu\nu}_\sigma(b) \frac{\partial}{\partial \alpha^\mu} \wedge \frac{\partial}{\partial \alpha^\nu} + \rho^{\mu i}(b) \frac{\partial}{\partial \alpha^\mu} \wedge \frac{\partial}{\partial b^i},$$

where $\rho^{\mu i}$ and $f^{\mu\nu}_\sigma$ are the anchor and structure functions respectively.

2.4. **Symplectic groupoids.** If G is a Lie groupoid, there are three natural maps from $G^{(2)}$ to G: the two projections p_1 and p_2 and the multiplication m. A differential form ω on G is said to be multiplicative if it satisfies the cocycle condition

$$m^*\omega = p_1^*\omega + p_2^*\omega.$$

(If ω is a function, this simply means $\omega(uv) = \omega(u) + \omega(v)$ for all $u, v \in G^{(2)}$.)

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[3]The corresponding differential on sections of the exterior algebra of the dual of $T^*_\pi M$ (i.e., on multivector fields) is the inner derivation $\delta := [\pi, \ ]_{\mathrm{SN}}$ where $[\ ,\ ]_{\mathrm{SN}}$ is the Schouten–Nijenhuis bracket. So $(T^*_\pi M, TM)$ is an example of Lie bialgebroid.

A **symplectic groupoid** [6, 9, 20] $(G \rightrightarrows M, \omega)$ is then by definition a Lie groupoid G over M endowed with a multiplicative symplectic form ω. It follows [6] that $\epsilon \colon M \hookrightarrow G$ is a Lagrangian embedding, that the inverse map is an anti-symplectomorphism, and that the base manifold M has a unique Poisson structure π such that the source and the target maps are Poisson and anti-Poisson respectively. The Lie algebroid of $(G \rightrightarrows M, \omega)$ turns then out to be isomorphic to $T_\pi^* M$.

It is proved in [12] that, given an integrable Poisson manifold (M, π), it is alway possible to endow a ssc Lie groupoid G such that $\mathsf{Lie}(G) = T_\pi^* M$ with a multiplicative symplectic form such that the induced Poisson structure on M is π. This means that the problem of integrating a Poisson manifold to its symplectic groupoid actually amounts just to integrating its Lie algebroid.

We discuss one single example of symplectic groupoid, which is relevant for the rest of the paper. Let $G \rightrightarrows B$ be a Lie groupoid. Endow $T^* G$ with the canonical symplectic structure ω. Let $A = \mathsf{Lie}(G)$; then one can endow $T^* G$ with a Lie groupoid structure with base A^* such that ω is multiplicative and the Poisson structure on A^* is the one described in 2.3.2 (see [6]). Moreover, if G is ssc, then so is $T^* G$. This shows that A^* is integrable as a Poisson manifold if A is integrable as a Lie algebroid. The converse is also true by Lie I on page 5 since A may be regarded as a Lie subalgebroid of $T^* A^*$ (see Lemma 4.2).

Thus, in order to integrate a Lie algebroid A, one may first look for the ssc symplectic groupoid of A^* and the look for the Lagrangian Lie subgroupoid whose Lie algebroid is A. This is not immediate as the ssc symplectic groupoid of A^* may not be presented as $T^* G$. In Sect. 4 we will however describe a method to do this explicitly.

2.5. **Symplectic reduction.** Let (S, ω) be a (possibly weak) symplectic manifold (see footnote 2) and C a submanifold. The orthogonal tangent bundle $T^\perp C$ is defined as the subbundle of $T_C S$ consisting of vectors that are ω-orthogonal to vectors tangent to C. The submanifold C is called **coisotropic** if $T^\perp C \subset TC$ and—as a particular case—**Lagrangian** if $T^\perp C = TC$. Since ω is closed, the subbundle $T^\perp C$ defines an involutive distribution (the **characteristic foliation**) on the coisotropic submanifold C whose leaves are exactly the kernel of ω. This implies that the leaf space $\underline{C}$, if smooth, is naturally endowed with a symplectic form $\underline{\omega}$ whose pullback to C is the restriction of ω.

If C is coisotropic, L is Lagrangian and their intersection is **clean** (viz., $L \cap C$ is also a submanifold and $T(L \cap C) = TL \cap TC$), then the image of the projection of $L \cap C$ to $\underline{C}$ is Lagrangian in $\underline{C}$ (it may not be an embedding but just an immersion), see [17, Lecture 3]. For this to

hold in infinite dimensions, one has to add explicitly a further condition (which in finite dimensions is automatically satisfied): $T^\perp(L \cap C) = TL + T^\perp C$. When this condition is satisfied we say that the intersection is symplectically regular.

Even when $\underline{C}$ is not smooth, we may think of it as a singular symplectic manifold and of the projection of $L \cap C$ as a singular Lagrangian submanifold.

3. Integration of Poisson manifolds

We briefly recall the method of [3] and fix the notations. Let M be a finite-dimensional Poisson manifold with Poisson bivector π. Let $PM := \{I \to M\}$ be the path space of M and T^*PM the manifold[4] of bundle maps $TI \to T^*M$. The fiber over $X \in PM$ may be identified with the space of sections $\Gamma(T^*I \otimes X^*T^*M)$. The canonical symplectic form Ω at a point $X \in PM$, $\eta \in T_X^*PM$ is defined by

$$\Omega(X, \eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \int_I \langle\, e_1\,,\, \xi_2\,\rangle - \langle\, e_2\,,\, \xi_1\,\rangle,$$
$$\xi_i \in \Gamma(X^*TM) = T_X PM, \quad e_i \in \Gamma(T^*I \otimes X^*T^*M) = T_X^*PM,$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between tangent and cotangent fibers of M. The submanifold $\mathcal{C}(M)$ defined as the space of solutions to the equations[5]

$$(3.1) \qquad\qquad\qquad \mathrm{d}X = \pi^\#(X)\eta$$

is coisotropic in T^*PM. Its characteristic foliation turns out to be expressed in terms of an infinitesimal action of the Lie algebra

$$\Gamma_0(M) := \{C \colon I \to \Omega^1(M) \mid C(0) = C(1) = 0\},$$

where the Lie bracket is defined pointwise in terms of the Lie bracket on $\Omega^1(M)$. To describe this action, it is easier to pass to local coordinates $\{x^i\}_{i=1,\ldots,\dim M}$ on M. Then the vector field on $\mathcal{C}(X)$ associated to an element C of $\Gamma_0(M)$ evaluated at a point (X, η) may be written as $(\delta X^i, \delta\eta_i)$ with

$$(3.2a) \qquad\qquad \delta X^i = -\pi^{ij}(X)\,(C_X)_j,$$
$$(3.2b) \qquad\qquad \delta\eta_i = -\mathrm{d}(C_X)_i - \partial_i\pi^{rs}\,(C_X)_r\,\eta_s,$$

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[4]A Banach manifold structure may be introduced by restricting to maps with a given degree of differentiability. For example, as in [3], one may define T^*PM as the space of continuous bundle maps with C^1-base maps.

[5]As observed in [15, 7], solutions to (3.1)—called "cotangent paths" in [8]—are precisely Lie algebroid morphisms $TI \to T_\pi^*M$, where TI is given its canonical Lie algebroid structure.

where C_X is the section of X^*T^*M defined by $C_X(t) = C(t)(X(t))$ and $\partial_i \pi^{rs}$ are the structure functions of the Lie algebroid $T^*_\pi M$ (w.r.t. the local basis $\{\mathrm{d}x^i\}$ of sections). Upon using the constraint equations (3.1), this local coordinate expression is well-defined. The leaf space $\underline{\mathcal{C}}(M)$ of $\mathcal{C}(M)$ is then the (possibly singular) ssc symplectic groupoid of M.

4. Integration of A and A^*

We now apply the method recalled in Sect. 3 to the Poisson manifold A^*, where A is a Lie algebroid. We denote by $\mathcal{P}(A)$ the manifold of bundle maps $TI \to A$. The central result of this paper is the following

Theorem 4.1. *$\mathcal{P}(A)$ is a Lagrangian submanifold of T^*PA^*, and the projection $\mathsf{G}(A)$ of $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ to $\underline{\mathcal{C}}(A^*)$ yields the (possibly singular) ssc Lie groupoid of A as a Lagrangian Lie subgroupoid of the symplectic groupoid of A^*.*

The rest of the Section is devoted to the proof of this Theorem.

Proof. We begin with an easy Lemma whose proof is left to the reader.

Lemma 4.2. *Let $A \to B$ be a vector bundle. The fiber of T^*A^* over a point $(b, \alpha) \in A$ is the vector space $T^*_b B \oplus A_b$. The map*

$$
\begin{array}{cccc}
\iota: & A & \to & T^*A^* \\
 & (b, a) & \mapsto & ((b, 0), 0 \oplus a)
\end{array}
$$

is an injective bundle map from $\begin{array}{c} A \\ \downarrow \\ B \end{array}$ *to* $\begin{array}{c} T^*A^* \\ \downarrow \\ A^* \end{array}$.

If T^*A^* is given the canonical symplectic structure, then ι is a Lagrangian embedding.

If A is a Lie algebroid and T^*A^* is given the Lie algebroid structure induced by the Poisson structure on A^*, then ι is a morphism of Lie algebroids. So A is a Lagrangian Lie subalgebroid of T^*A^*.

As a consequence, the composition of a bundle map $TI \to A$ with ι yields a bundle map $TI \to T^*A^*$. So ι induces an inclusion of $\mathcal{P}(A)$ into T^*PA^* which is also Lagrangian.

Lemma 4.3. *The intersection $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ consists of Lie algebroid morphisms $TI \to A$.*

Proof. Choosing local coordinates as in Sect. 2.1, we denote an element of T^*PA^* by the functions X^i, α^μ together with the 1-forms η_i and a_μ (observe that at different points in I we may be on different patches of

local coordinates). Using the Poisson bivector field π defined in (2.2), the constraint equations (3.1) read

$$\text{(4.1a)} \qquad \mathrm{d}X^i = \rho^{\mu i}(X)a_\mu,$$

$$\text{(4.1b)} \qquad \mathrm{d}\alpha^\mu = -\rho^{\mu i}(X)\eta_i - \alpha^\sigma f_\sigma^{\mu\nu}(X)a_\nu.$$

Elements of $\mathcal{P}(A)$ are represented by functions and 1-forms as above with the conditions $\alpha^\mu = 0$ and $\eta_i = 0\ \forall \mu, i$. So the intersection consists of functions X^i together with 1-forms a_μ satisfying the first equation above, which is precisely the local coordinate expression of the condition that the bundle map $TI \to A$ defined by (X, a) is a morphism of Lie algebroids. Observe finally that both sides of (4.1b) identically vanish, so no further conditions are imposed. $\qquad\square$

In the following we will denote by $\mathsf{P}(A)$ the space of Lie algebroid morphisms $TI \to A$. Given a second unit interval $J = [0, 1]$, we denote by $\mathsf{P}_2(A)$ the space of Lie algebroid morphisms $T(I \times J) \to A$. Two elements γ_0 and γ_1 of $\mathsf{P}(A)$ are said to be Lie algebroid homotopic, if there exists an element of $\mathsf{P}_2(A)$ that restricts to γ_u at $TI \times \{u\}$, $u = 0, 1$, and is trivial at $\{v\} \times TJ$, $v \in \partial I$.

Lemma 4.4. *The foliation of $\mathcal{C}(A^*)$ restricted to $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ is precisely the infinitesimal version of Lie algebroid homotopies.*

Proof. The foliation is defined by (3.2). Given a map $X\colon I \to B$, an element C of the Lie algebra $\Gamma_0(A^*)$ yields a section C_X of $X^*T^*A^* \to I$. Again choosing local coordinates, we may denote the components of $C_X(t)$ in $T^*_{X(t)}B$ by $\beta_i(t)$ and those in $A_{X(t)}$ by $b_\mu(t)$, $t \in I$. The vector field corresponding to (β, b) evaluated at a point (X, α, η, a) may be written using (3.2) and (2.2) as $(\delta X^i, \delta\alpha^\mu, \delta\eta_i, \delta a_\mu)$ with

$$\delta X^i = -\rho^{\mu i}(X)b_\mu,$$
$$\delta\alpha^\mu = \rho^{\mu i}(X)\beta_i + \alpha^\sigma f_\sigma^{\mu\nu}(X)b_\nu,$$
$$\delta\eta_i = -\mathrm{d}\beta_i - \alpha^\sigma \partial_i f_\sigma^{\mu\nu}(X)a_\mu b_\nu - \partial_i \rho^{\mu j}(X)(a_\mu\beta_j - \eta_j b_\mu),$$
$$\delta a_\mu = -\mathrm{d}b_\mu - f_\mu^{\nu\sigma}(X)a_\nu b_\sigma.$$

The restriction of this foliation to $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ is given by the first and last equations (the remaining ones are automatically satisfied when we impose $\alpha = \eta = 0$ and, consequently, $\delta\alpha = \delta\eta = 0$). If we integrate the flow of this vector field on a time-interval $(-\epsilon, \epsilon)$, these equations are then precisely the local coordinate expressions of a morphism of Lie algebroids $T(I \times (-\epsilon, \epsilon)) \to A$. $\qquad\square$

This shows that the projection of $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ to $\underline{\mathcal{C}}(A^*)$ is equal to the quotient $\mathsf{G}(A)$ of $\mathsf{P}(A)$ by Lie algebroid morphisms. It was shown

in [7], along the lines of [3], that $\mathsf{G}(A)$ has a groupoid structure and that, if it is smooth, has A as its Lie algebroid.

Lemma 4.5. *The intersection of $\mathcal{P}(A)$ and $\mathcal{C}(A^*)$ is clean and symplectically regular.*

Proof. The intersection $\mathcal{P}(A) \cap \mathcal{C}(A^*)$ may be given a manifold structure, see [7]. An element (X, a) in it is a solution to (4.1a). So a tangent vector at (X, a) is a pair $(\dot{X}, \dot{a})$ satisfying

(4.2) $$\mathrm{d}\dot{X}^i = \rho^{\mu i}(X)\dot{a}_\mu + \dot{X}^j \partial_j \rho^{\mu i}(X) a_\mu.$$

On the other hand, the tangent vector at $(X, \alpha, \eta, a) \in \mathcal{C}(A^*)$ consists of a quadruple $(\dot{X}, \dot{\alpha}, \dot{\eta}, \dot{a})$ satisfying

$$\mathrm{d}\dot{X}^i = \rho^{\mu i}(X)\dot{a}_\mu + \dot{X}^j \partial_j \rho^{\mu i}(X) a_\mu,$$
$$\mathrm{d}\dot{\alpha}^\mu = -\rho^{\mu i}(X)\dot{\eta}_i - \dot{X}^j \partial_j \rho^{\mu i}(X) \eta_i +$$
$$- \alpha^\sigma f_\sigma^{\mu\nu}(X)\dot{a}_\nu - \dot{\alpha}^\sigma f_\sigma^{\mu\nu}(X) a_\nu - \alpha^\sigma \dot{X}^j \partial_j f_\sigma^{\mu\nu}(X) a_\nu.$$

At an intersection point with $\mathcal{P}(A)$ we have to set $\alpha = 0$ and $\eta = 0$, and intersecting with $T\mathcal{P}(A)$ means setting $\dot{\alpha} = 0$ and $\dot{\eta} = 0$; so we recover (4.2). This shows that the intersection in clean.

Elements of $T\mathcal{P}(A) + T\mathcal{C}(A^*)$ at a point $(X, 0, 0, a) \in \mathcal{P}(A) \cap \mathcal{C}(A^*)$ are vectors of the form $(\dot{X}, \delta\alpha, \delta\eta, \dot{a})$ where $\dot{X}$ and $\dot{a}$ are arbitrary while

$$\delta\alpha^\mu = \rho^{\mu i}(X)\beta_i,$$
$$\delta\eta_i = -\mathrm{d}\beta_i - \partial_i \rho^{\mu j}(X) a_\mu \beta_j.$$

An explicit, though lengthy, computation shows that these are precisely all possible vectors in $T^\perp(\mathcal{P}(A) \cap \mathcal{C}(A^*))$. $\qquad\square$

By the discussion in 2.5 we conclude the proof of the Theorem. $\quad\square$

If we recall that the Lie groupoid of A must appear as a Lagrangian Lie subgroupoid of the symplectic groupoid of A^* (see the end of 2.4), we may interpret the above result as a way of deriving the method of [15, 7] from the one of [3].

Finally observe that, with the above notation, $\mathcal{C}(M) = \mathsf{P}(T_\pi^* M)$ and $\underline{\mathcal{C}}(M) = \mathsf{G}(T_\pi^* M)$, so the method of [3] may also be recovered as a particular case of [15, 7].

5. INTEGRATION OF COISOTROPIC SUBMANIFOLDS

Let (M, π) be a Poisson manifold. A submanifold C is called coisotropic if $\pi^\#(N^*C) \subset TC$, where N^*C denotes the conormal bundle of C (viz., the subbundle of $T_C^* M$ of covectors that vanish when applied to a vector tangent to C). In case the Poisson structure of M comes from

a strong symplectic structure, this definition coincides with the usual one recalled in 2.5 since, in this case, $\pi^{\#}$ establishes an isomorphism between N^*C and $T^{\perp}C$.

The theory of coisotropic submanifolds of Poisson manifolds [18] generalizes many properties of the corresponding theory in the symplectic case (e.g., one may generalize symplectic reduction to the Poisson case as it turns out that $\pi^{\#}(N^*C)$ is an integrable distribution on the coisotropic submanifold C and that the leaf space inherits a Poisson structure). Moreover, coisotropic submanifolds label the possible boundary conditions of the Poisson sigma model yielding the beginning of a theory of quantum reduction in the deformation quantization context [4].

From the point of view of the present paper, coisotropic submanifolds are important because of their relations with the theory of Lie algebroids. Namely:

Proposition 5.1. *A submanifold C of (M, π) is coisotropic iff N^*C is a Lie subalgebroid of T_π^*M.*

Proposition 5.2. *Let A be a Lie algebroid and A^* its dual regarded as a Poisson manifold. Then the zero section of A^* is coisotropic and its conormal bundle is the inclusion ι of A as a Lagrangian Lie subalgebroid of T^*A^* described in Lemma 4.2.*

Proof of Prop. 5.1. If N^*C is a Lie subalgebroid, in particular its anchor $N^*C \to TC$ is the restriction of $\pi^{\#}$ to N^*C; this immediately shows that C is coisotropic.

The converse is true by Corollary 3.1.5 in [18], but for completeness we give a proof here. Assume that C is coisotropic. By definition, the restriction of $\pi^{\#}$ to N^*C maps it to TC and so it defines an anchor. It remains only to prove that the Koszul bracket induces a bracket on sections of N^*C. Let U be a trivializing chart on M intersecting C. We choose adapted local coordinates $\{x^I\}_{I=1,\ldots,\dim M}$ so that $U \cap C$ is determined by $x^I = 0$, $I = \dim C + 1, \ldots, \dim M$. To make the notation more transparent, we will use small Latin indices to denote the first $\dim C$ coordinates (the tangential ones) and small Greek indices to denote the remaining (transversal) ones; when we do not want to distinguish them, we will use capital Latin indices. So the above conditions may be written $x^\mu = 0$. If C is coisotropic, then $\pi^{\mu\nu}(x) = 0$ for all $x \in U \cap C$, and as a consequence $\partial_i \pi^{\mu\nu}(x) = 0$ for all $x \in U \cap C$. This implies that

$$[\,\mathrm{d}x^\mu\,,\,\mathrm{d}x^\nu\,](x) = \partial_K \pi^{\mu\nu}(x)\,\mathrm{d}x^K = \partial_\mu \pi^{\mu\nu}(x)\,\mathrm{d}x^\mu, \qquad \forall x \in U \cap C.$$

This concludes the proof. $\qquad\square$

Proof of Prop. 5.2. Using coordinates (b, α) as in 2.3.2, we see that on the zero section $\alpha = 0$ only the mixed components of the bivector field in (2.2) survive. This shows that the zero section is coisotropic. Its conormal bundle consists of elements in T^*A^* of the form $((b, 0), 0 \oplus a)$, so it is $\iota(A)$. $\qquad \square$

Observe that the conormal bundles of submanifolds of M are all possible Lagrangian subbundles of T^*M with its canonical symplectic structure. So Prop. 5.1 may also be rephrased as

Proposition 5.3. *The set of coisotropic submanifolds of M is isomorphic to the set of Lagrangian Lie subalgebroids of $T^*_\pi M$.*

In [19, Sect. 4] coisotropic subgroupoids of Poisson–Lie groupoids are studied. It follows from Prop. 4.10 there, as a particular case, that the Lie algebroid of a Lagrangian Lie subgroupoid of the symplectic groupoid of the Poisson manifold (M, π) is a Lagrangian Lie subalgebroid of $T^*_\pi M$. We want to prove that also the converse of this statement is true.

Theorem 5.4. *Let (M, π) be an integrable Poisson manifold and $\underline{\mathcal{C}}(M)$ its ssc symplectic groupoid. Then there is a one-to-one correspondence between Lagrangian Lie subgroupoids of $\underline{\mathcal{C}}(M)$ and coisotropic submanifolds of M.*

One direction of the isomorphism follows from the cited Prop. 4.10 of [19] together with Prop. 5.3 above. We have then to construct an inverse map from coisotropic submanifolds of M to Lagrangian Lie subgroupoids of $\underline{\mathcal{C}}(M)$. We do it using the technique of [3] recalled in Sect. 3 and actually prove a more general statement:

Proposition 5.5. *To each coisotropic submanifold C of M there corresponds a (possibly singular) Lagrangian Lie subgroupoid (isomorphic to $\mathsf{G}(N^*C)$ as a groupoid) of $\underline{\mathcal{C}}(M)$.*

Observe then that, thanks to Prop. 5.2, Thm. 4.1 is now a particular case of this Proposition. To prove the Proposition we introduce

(5.1) $\mathcal{L}(C) := \{(X, \eta) \in T^*PM : X \in PC, \ \eta \in \Gamma(T^*I \otimes X^*N^*C)\}.$

It is easy to see that $\mathcal{L}(C)$ is a Lagrangian submanifold of T^*PM and that $\mathcal{L}(C) \cap \mathcal{C}(C)$ is the manifold $\mathsf{P}(N^*C)$ of Lie algebroid morphisms $TI \to N^*C$. To complete the proof of Prop. 5.5 (and hence of Thm. 5.4), by the discussion in 2.5 we only need the following two Lemmata:

Lemma 5.6. *The restriction to $\mathcal{L}(C) \cap \mathcal{C}(C)$ of the characteristic foliation of $\mathcal{C}(C)$ is the infinitesimal form of Lie algebroid homotopies $\mathsf{P}_2(N^*C)$.*

Lemma 5.7. *The intersection $\mathcal{L}(C) \cap \mathcal{C}(C)$ is clean and symplectically regular.*

Proof of Lemma 5.6. For simplicity we use adapted local coordinates as in the proof of Prop. 5.1 (again observe that at different points in I we may be on different patches of adapted local coordinates). The intersection of the foliation (3.2) with $\mathcal{L}(C)$ amounts to the constraints $\delta X^\mu = 0$ and $\delta \eta_i = 0$. On $\mathcal{L}(C)$ we have $\eta_i = 0$ and $\pi^{\mu\nu}(X) = 0$; so

$$\delta \eta_i = -\mathrm{d}(C_X)_i - \partial_i \pi^{RS} \eta_R \, (C_X)_S = -\mathrm{d}(C_X)_i - \partial_i \pi^{\mu k} \eta_\mu \, (C_X)_k,$$

and the condition $\delta \eta_i = 0$ together with the boundary conditions on C_X implies $(C_X)_i = 0$. Then $\delta X^\mu = -\pi^{\mu j}(X)(C_X)_j$ automatically vanishes. Moreover, we get

$$(5.2) \qquad \delta X^i = -\pi^{i\mu}(X)\,(C_X)_\mu,$$

$$(5.3) \qquad \delta \eta_\mu = -\mathrm{d}(C_X)_\mu - \partial_\mu \pi^{\nu\tau}\,(C_X)_\nu\,\eta_\tau.$$

The local flow of this vector field on a time interval $(-\epsilon, \epsilon)$ precisely defines a Lie algebroid morphism $T(I \times (-\epsilon, \epsilon)) \to N^*C$. $\qquad\square$

Proof of Lemma 5.7. As already recalled $\mathcal{L}(C) \cap \mathcal{C}(C) = \mathsf{P}(N^*C)$ is a manifold, see [7]. Let now $(X, \eta) \in \mathcal{L}(C)$. Using again adapted local coordinates, we see that a vector $(\dot{X}, \dot{\eta}) \in T_{(X,\eta)}T^*PM$ belongs to $T_{(X,\eta)}\mathcal{C}(M)$ iff

$$\mathrm{d}\dot{X}^I = \pi^{IK}(X)\,\dot{\eta}_K + \dot{X}^J \partial_J \pi^{I\nu}(X)\,\eta_\nu,$$

where we have used $\eta_i = 0$. The intersection of $T_{(X,\eta)}\mathcal{L}(C)$ with $T_{(X,\eta)}\mathcal{C}(M)$ is then determined by also imposing the equations $\dot{X}^\mu = \dot{\eta}_i = 0$ (which express belonging to $T_{(X,\eta)}\mathcal{L}(C)$); viz.,

$$\mathrm{d}\dot{X}^i = \pi^{i\nu}(X)\,\dot{\eta}_\nu + \dot{X}^j \partial_j \pi^{i\nu}(X)\,\eta_\nu,$$

$$\mathrm{d}\dot{X}^\mu = \pi^{\mu\nu}(X)\,\dot{\eta}_\nu + \dot{X}^j \partial_j \pi^{\mu\nu}(X)\,\eta_\nu.$$

The first equation says that $(\dot{X}^i, \dot{\eta}_\mu)$ belongs to $T_{(X,\eta)}(\mathcal{L}(C) \cap \mathcal{C}(C))$. The condition $\dot{X}^\mu = 0$ implies $\mathrm{d}\dot{X}^\mu = 0$, but thanks to $\pi^{\mu\nu}(X) = 0$ and $\partial_j \pi^{\mu\nu}(X) = 0$ the second equation set to zero does not put extra conditions on $(\dot{X}^i, \dot{\eta}_\mu)$. This shows that the intersection is clean.

An explicit but lengthy computation shows that the intersection is also symplectically regular and in particular that the elements of

$$T_{(X,\eta)}(\mathcal{L}(C) \cap \mathcal{C}(C)) = T_{(X,\eta)}\mathcal{L}(C) + T_{(X,\eta)}\mathcal{C}(C),$$

$(X, \eta) \in \mathcal{L}(C) \cap \mathcal{C}(C)$, are of the form $(\dot{X}^i, \delta X^\mu, \delta\eta_i, \dot{\eta}_\mu)$ with

$$\delta X^\mu = -\pi^{\mu j}(X)\,(C_X)_j,$$
$$\delta\eta_i = -\mathrm{d}(C_X)_i - \partial_i \pi^{r\nu}\,(C_X)_r\,\eta_\nu,$$

and $\dot{X}^i$ and $\dot{\eta}_\mu$ arbitrary. $\qquad\qquad\qquad\square$

6. The twisted case

A **twisted symplectic manifold** is a manifold endowed with a nondegenerate 2-form. A **twisted Poisson manifold** (M, π, ϕ) is a manifold M endowed with a bivector field π and a closed 3-form ϕ such that

$$[\,\pi\,,\,\pi\,] = \frac{1}{2}\,\wedge^3 \pi^\#\phi.$$

One may still define a bracket on functions by $\{\,f\,,\,g\,\} = \pi(\mathrm{d}f, \mathrm{d}g)$, but the Jacobi identity will not be satisfied. A twisted strong symplectic manifold (M, ω) provides an example of twisted Poisson manifold by setting π to be the inverse of ω and $\phi = \mathrm{d}\omega$. The cotangent bundle of a twisted Poisson manifold is a Lie algebroid with $\pi^\#$ as its anchor and a Lie bracket that on exact 1-forms reads $[\,\mathrm{d}f\,,\,\mathrm{d}g\,] = \mathrm{d}\{\,f\,,\,g\,\} + \iota_{\pi^\#\mathrm{d}f}\iota_{\pi^\#\mathrm{d}g}\phi$. We will denote this Lie algebroid by $T^*_{\pi,\phi}M$.

A **twisted symplectic groupoid** $(G \rightrightarrows M, \omega, \phi)$ is a Lie groupoid $G \rightrightarrows M$ endowed with a nondegenerate, multiplicative 2-form ω on G and with a closed 3-form ϕ on M such that the cocycle condition $\mathrm{d}\omega = s^*\phi - t^*\phi$ holds. It turns out [5] that a twisted Poisson structure (π, ϕ) is induced on the base M such that $\mathsf{Lie}(G)$ is isomorphic to $T^*_{\pi,\phi}M$.

Symplectic reduction may be generalized to the twisted case. Let (S, ω) be a twisted symplectic manifold. A submanifold C is called **coisotropic** if $T^\perp C \subset TC$ and the restriction ω_C of ω to C is invariant (viz., $\mathrm{L}_X\omega_C = 0$ for any $X \in \Gamma(T^\perp C)$). A **Lagrangian** submanifold is a submanifold L such that $T^\perp L = TL$, and it is automatically coisotropic. It turns out that $T^\perp C$ defines a foliation on the coisotropic submanifold C and that the leaf space $\underline{C}$ inherits a twisted symplectic structure (if smooth). Moreover, if L is Lagrangian and the intersection $L \cap C$ is clean and symplectically regular, the image of the projection of $L \cap C$ to $\underline{C}$ is also Lagrangian.

We now recall the method introduced in [5] to integrate twisted Poisson manifolds to (possibly singular) twisted symplectic groupoids by modifying the method of [3]. One considers again the submanifold $\mathcal{C}(M)$ of T^*PM as in Sect. 3. This is not coisotropic w.r.t. the canonical symplectic form Ω of T^*PM, but it is so w.r.t. the twisted symplectic

form $\widetilde{\Omega} := \Omega + \widehat{\Omega}$ with

$$\widehat{\Omega}(X,\eta)(\xi_1 \oplus e_1, \xi_2 \oplus e_2) = \frac{1}{2}\int_I \phi(X)(\pi^\#(X)\eta, \xi_1, \xi_2),$$

$\xi_1, \xi_2 \in \Gamma(X^*TM)$, $e_1, e_2 \in \Gamma(T^*I \otimes X^*T^*M)$. The twisted symplectic groupoid of M turns then out to be the leaf space $\underline{\mathcal{C}}(M)$.

Finally, we want to generalize the results of Sect. 5. We say that a submanifold C is coisotropic in the twisted Poisson manifold (M, π, ϕ) if $\pi^\#(N^*C) \subset TC$ and the restriction ϕ_C of ϕ to C is horizontal (viz., $\iota_X \phi_C = 0$ for any $X \in \Gamma(N^*C)$). If the twisted Poisson structure comes from a twisted strong symplectic structure, then this definition coincides with the one given in the twisted symplectic case.

One may easily see (along the lines of Sect. 5) that, if C is coisotropic, N^*C is a Lie subalgebroid of $T^*_{\pi,\phi}M$ and that the leaf space $\underline{C}$ inherits the structure of a twisted Poisson manifold. However, N^*C might be a Lie subalgebroid of $T^*_{\pi,\phi}M$ in more general instances. Thus, we no longer have a one-to-one correspondence between coisotropic submanifolds of (M, π, ϕ) and Lagrangian Lie subalgebroids of $T^*_{\pi,\phi}M$ if on T^*M we put the canonical symplectic structure $\omega = \mathrm{d}p_i\,\mathrm{d}x^i$. On the other hand, if we put on T^*M the twisted symplectic structure[6] $\widetilde{\omega} := \omega + \widehat{\omega}$ with $2\widehat{\omega} = p_i\pi^{ij}\phi_{jkl}\,\mathrm{d}x^k\,\mathrm{d}x^l$, it turns out that N^*C is Lagrangian in $(T^*M, \widetilde{\omega})$ iff ϕ_C is horizontal; so Prop. 5.3 generalizes to the twisted case, provided one twists the symplectic form on T^*M. One may finally generalize Prop. 5.5 to the twisted case since $\mathcal{L}(C)$, defined as in (5.1), turns out to be Lagrangian in $(T^*PM, \widetilde{\Omega})$ if C is coisotropic, and it is possible to show Lemmata 5.6 and 5.7 in this case (with very similar proofs).

A further modification of the integration methods of [3, 5] is considered in [1] to integrate (twisted) Dirac structures, a common generalization of (twisted) symplectic and Poisson structures. It would be interesting to know if any of the ideas in the present paper have a generalization in that context.

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[6]In [5] it is observed that $(T^*_{\pi,\phi}M, TM)$ is actually an example of quasi Lie bialgebroid (cf. footnote 3 on page 6) as there is a derivation δ of $\Omega^\bullet(M)$ (which deforms the exterior derivative) such that

$$\delta[\sigma, \tau] = [\delta\sigma, \tau] + [\sigma, \delta\tau], \qquad \forall\sigma, \tau \in \Omega^1(M),$$

and that $\delta^2 = [\phi, \bullet]$ (where we have extended the Lie bracket to the whole of $\Omega^\bullet(M)$ as a biderivation). This corresponds to having a quasi (i.e., no Jacobi) Lie algebroid structure on TM that is compatible with the one on $T^*_{\pi,\phi}M$. This quasi Lie algebroid structure determines a twisted Poisson structure on T^*M, and as this turns out to be nondegenerate, it corresponds to a twisted symplectic structure that is precisely $\widetilde{\omega}$.

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